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# Two-Loop Static QCD Potential for General Colour State

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**Abstract**

In this letter, we extend the known results for the QCD potential between a static quark and its antiquark by computing the two-loop corrections to the colour-octet state.

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The QCD potential between a static quark and its antiquark has for a long time been used as a probe of the fundamental properties of the strong interactions such as asymptotic freedom and confinement [1]. Historically, the potential for a quark-antiquark pair in the colour-singlet state attracted the most attention because it is a basic ingredient in the theory of heavy quarkonium and, therefore, of primary phenomenological interest. Nowadays, however, there is growing interest in its colour-octet counterpart. The latter naturally appears in effective-theory calculations of high-order corrections to the heavy-quarkonium spectrum and decay rates through the so-called ultrasoft contribution [2]. Moreover, it determines the properties of glueballinos and is necessary for the analysis of gluino-antigluino threshold production [3,4]. It is also used in lattice QCD for studying the behavior of strong interactions at long distances and the interplay between perturbative and non-perturbative physics [3]. This requires knowledge of the corresponding perturbative corrections which, in contrast to the colour-singlet case, are not available

beyond one loop. In the present letter, we fill this gap and compute the $\mathcal{O}(\alpha_s^2)$ correction to the colour-octet static potential.

The perturbative expansion of the colour-singlet potential reads

$$V(|\boldsymbol{q}|) = -\frac{4\pi C_F \alpha_s(|\boldsymbol{q}|)}{\boldsymbol{q}^2}\left[1 + \frac{\alpha_s(|\boldsymbol{q}|)}{4\pi} a_1 + \left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^3 \left(a_3 + 8\pi^2 C_A^3 \ln\frac{\mu^2}{\boldsymbol{q}^2}\right) + \cdots\right], \tag{1}$$

where the first term corresponds to the Coulomb potential. The one-loop coefficient,

$$a_1 = \frac{31}{9} C_A - \frac{20}{9} T_F n_l, \tag{2}$$

has been known for a long time [5,6], while the two-loop coefficient, $a_2$, has only recently been found [7,8,9]. In Ref. [9], the result of Ref. [8] was confirmed,

$$a_2 = \left[\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3)\right] C_A^2 - \left[\frac{1798}{81} + \frac{56}{3}\zeta(3)\right] C_A T_F n_l - \left[\frac{55}{3} - 16\zeta(3)\right] C_F T_F n_l + \left(\frac{20}{9} T_F n_l\right)^2, \tag{3}$$

where $\zeta$ is Riemann's zeta function, with value $\zeta(3) = 1.202057\ldots$. Here, $C_A = N$ and $C_F = (N^2-1)/(2N)$ are the eigenvalues of the quadratic Casimir operators of the adjoint and fundamental representations of the $SU(N)$ colour gauge group, respectively, $T_F = 1/2$ is the index of the fundamental representation, and $n_l$ is the number of light-quark flavours. The modified minimal-subtraction ($\overline{\mathrm{MS}}$) scheme for the renormalization of $\alpha_s$ is implied. The logarithmic term of $\mathcal{O}(\alpha_s^3)$ in Eq. (1) reflects the infrared divergence of the static potential [10]. The particular form of the logarithmic term corresponds to dimensional regularization [11]. The corresponding infrared-divergent term is cancelled against the ultraviolet-divergent one of the ultra-soft contribution [2] in the calculation of the physical heavy-quarkonium spectrum [11,12]. The non-logarithmic third-order term, $a_3$, is still unknown.

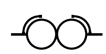
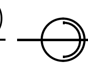
The perturbative expansion of the potential for the colour-octet state can be cast in the form

$$V^o(|\boldsymbol{q}|) = \frac{4\pi \alpha_s(|\boldsymbol{q}|)}{\boldsymbol{q}^2}\left(\frac{C_A}{2} - C_F\right)\left[1 + \frac{\alpha_s(|\boldsymbol{q}|)}{4\pi} a_1^o + \left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^2 a_2^o + \left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^3 \left(a_3^o + 8\pi^2 C_A^3 \ln\frac{\mu^2}{\boldsymbol{q}^2}\right) + \cdots\right], \tag{4}$$

where the one-loop coefficient is the same as in the colour-singlet case, $a_1^o = a_1$. The two-loop coefficient, however, differs by a finite renormalization-independent term,

$$a_2^o = a_2 + \delta a_2. \tag{5}$$

Our result is

$$\delta a_2 = C_A^2 \frac{3d-11}{d-5} \Bigg[$$

$$- \frac{3(d-4)(d-1)}{d-5}$$

$$\Bigg]$$
$$= \left(\pi^4 - 12\pi^2\right) C_A^2 + \mathcal{O}(d-4)\,, \tag{6}$$

where $d$ is the space-time dimension. The non-logarithmic part of the three-loop coefficient, $a_3^o$, is still unknown. It is instructive to look at the numerical size of the corrections. For $N = 3$ one obtains $\delta a_2 = -189.2$. At the same time, we have $a_2 = 155.8(211.1, 268.8)$ and $a_1 = 4.778(5.889, 7.000)$ for $n_l = 5(4, 3)$. Thus, in the colour-octet case, the two-loop correction is significantly smaller than for the colour-singlet configuration. Depending on $n_l$, it even changes sign.

In the remaining part of this letter, we wish to describe two independent ways that have been used to evaluate $\delta a_2$. The first method proceeds along the lines of the analysis [9,11] based on the threshold expansion [13]. In general, the threshold expansion is the proper framework for performing calculations involving a heavy quark-antiquark system. It provides rigorous power-counting rules and natural definitions of the formal expressions obtained in the perturbative analysis of the nonrelativistic effective theory. The corrections to the static potential only arise from the soft regions of the loop integrals, which are characterized by the following scaling of the loop momenta: $l_0 \sim |\boldsymbol{l}| \sim |\boldsymbol{q}|$. Thus, the calculation of the coefficients $a_i$ and $a_i^o$ can be performed in the static limit of NRQCD, $m_q \to \infty$.

Due to the exponentiation of the static potential [5], the coefficients $a_i$ of the colour-singlet state only receive contributions from the maximally non-Abelian parts, leaving aside the terms involving $n_l$. The selection of these parts effectively retains the contributions of the soft region, as the appearance of the Abelian colour factor $C_F$ indicates the presence of a Coulomb pinch and thus implies that at least one loop momentum is potential. The latter contributions just represent iterations of the lower-order potential and, therefore, should be excluded from the potential itself. In the nonrelativistic effective theory, these iterations are taken into account in the perturbative solution of the Schrödinger equation about the Coulomb approximation. These contributions refer to dynamical rather than static heavy-quark and -antiquark fields, and the Coulomb pinch singularities we encounter in the static-limit calculations are resolved by keeping a finite mass in the nonrelativistic heavy-quark propagator.

The analysis of the colour-octet state is more involved, since, in this case, the Coulomb pinches come with all possible colour factors and cannot be removed by selecting the maximum non-Abelian ones. Thus, the separation of the Coulomb pinches should be performed explicitly. They appear in the Feynman diagrams involving the product of the nonrelativistic quark and antiquark propagators,

$$\frac{1}{k_0 - \boldsymbol{k}^2/(2m_q) + i\varepsilon}\,\frac{1}{k_0 + \boldsymbol{k}^2/(2m_q) - i\varepsilon}. \tag{7}$$

In this case, after expanding the quark propagator in $1/m_q$, one obtains ill-defined products like

$$\frac{1}{(k_0+i\varepsilon)^m}\,\frac{1}{(k_0-i\varepsilon)^n}. \tag{8}$$

Thus, separating the soft and potential regions is unavoidable.[1] In the soft region, the pole contributions of the quark and antiquark propagators have to be excluded, and the product in Eq. (8) should actually be defined to be its principal value,

$$\frac{1}{2}\left[\frac{1}{(k_0+i\varepsilon)^{m+n}}+\frac{1}{(k_0-i\varepsilon)^{m+n}}\right]. \tag{9}$$

In the potential region, the quark and antiquark propagator poles produce contributions of the form

$$-\,i\pi\,\frac{m_q}{\boldsymbol{k}^2-i\varepsilon}\left[\delta\left(k_0-\frac{\boldsymbol{k}^2}{2m_q}\right)+\delta\left(k_0+\frac{\boldsymbol{k}^2}{2m_q}\right)\right], \tag{10}$$

where the $1/v$ Coulomb singularity shows up explicitly. After integration over $k_0$, Eq. (10) yields the nonrelativistic Green function of the free Schrödinger equation. Only Eq. (9) should be taken into account in the calculation of the static potential.

At one-loop, there is only one diagram involving a Coulomb pinch, namely, the planar box, which has the colour factor $C_F^2$ for the colour-singlet state. Picking up the soft contribution, i.e., using the principal-value prescription of Eq. (9) to define Eq. (8), we find the planar box to cancel the $C_F^2$ part of the non-planar box, which in total is proportional to $C_F^2-C_FC_A/2$. This explicitly demonstrates the exponentiation of the one-loop colour-singlet static potential in momentum space. However, we can also turn things around and express the planar box with Coulomb pinches through the well-defined non-planar box by actually requiring the cancellation of the $C_F^2$ terms in the sum of all one-loop diagrams, as is dictated by the exponentiation. The result for $a_1^o$ as given above is then obtained by simply replacing the colour-singlet colour factor by the colour-octet one.

This strategy carries over to two loops. Here, we have diagrams with zero, one, or two Coulomb pinches. For the diagrams without Coulomb pinch, the contribution to $a_2^o$ is obtained by adopting the correct colour factor. We divide the Feynman diagrams with Coulomb pinches into those that have two quark and two antiquark propagators (cf. Fig. 1) and the rest. The latter ones are treated directly using the principal-value prescription of Eq. (9). For the former, however, it is simpler to use the exponentiation, which requires that the diagrams contributing to the colour factors $C_AC_F^2$ and $C_F^3$ sum up to zero in the colour-singlet case. This leads to two equations for the diagrams suffering from Coulomb pinches, namely, those shown in Figs. 1a and b, which can be solved. This provides a result in terms of the diagrams in Figs. 1c and d, which are free of pinches. After

[1] Note that, for the diagrams without Coulomb pinches, the separation of the soft and potential regions is ambiguous and even gauge dependent. In such diagrams, the nonrelativistic quark and antiquark propagators can be safely expanded in $1/m_q$.

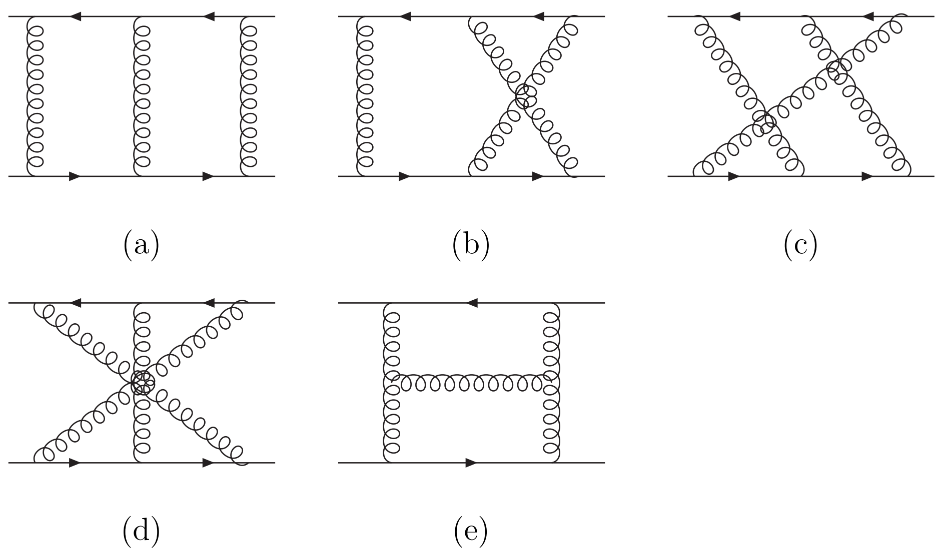



Figure 1: Two-loop Feynman diagrams with ((a) and (b)) and without ((c), (d), and (e)) Coulomb pinches that contribute to $\delta a_2$.

adopting the colour factors corresponding to the colour-octet configuration, one obtains the contributions to the results given in Eqs. (5) and (6). We wish to mention that the calculation was performed in the general covariant gauge and that the dependence on the gauge parameter was found to cancel out in the final result.

The second method to compute $V^o$ proceeds along the lines of Ref. [8]. While in the above, we had to assume exponentiation of the colour-singlet potential, we will now relax that assumption. The reason is that, although exponentiation is plausible to all orders of perturbation theory, the proof given in Ref. [5] holds for the singlet potential in *Abelian* theories only.

As a starting point, we now expand the logarithm of the $(T \times R)$ Wilson loop spanned by the static quark-antiquark pair at distance $R$ through $\mathcal{O}(\alpha_s^3)$. Taking the limit $T \to \infty$ (which, in a diagrammatic sense, 'cuts' the Wilson loop twice and restores translational invariance in the temporal direction, hence guaranteeing energy conservation at the vertices and leading to simple momentum-space Feynman rules) and inserting $SU(N)$ generators $T^a$ into the purely spatial Wilson lines to obtain the colour-octet potential to this order (for a manifestly gauge-invariant definition, see Ref. [14]), we now explicitly keep disconnected as well as one-particle-reducible diagrams in our expansion.

At this point, the general structure of the expansion involves (products of) up to two-loop four-point functions of static quarks (cf. Fig. 1). After Fourier transforming to momentum space, we can choose a special point to evaluate these four-point functions, since the potential, of course, only knows about the distance $R$ of the $q\bar{q}$ pair, which in a

momentum space representation translates into the momentum transfer $|\boldsymbol{q}|$ between the upper and lower lines in Fig. 1. Hence, effectively, we have to compute two-point functions with external static quarks, external momentum $q = (0, \boldsymbol{q})$, and internal static quarks, gluons, ghosts and light quarks, with the additional occurrence of a static (anti-) quark – gluon two-point vertex, resulting from the special kinematics.

After performing the colour algebra and exploiting symmetries of the integrals occurring in the expansion, all integrals which might give rise to pinch singularities, and had to be treated with caution in our first approach, cancel exactly. Thus, we are left with the task of computing a class of two-loop two-point integrals for which there exists a generic algorithm [8,15], based on integration by parts (IBP) [16]. The implementation in Ref. [15] (see also Chapter 6 of Ref. [17]) is based on Ref. [18].

Having generated the relevant set of diagrams and reduced the occurring Feynman integrals to the set of two-point functions described above, we now employ the reduction algorithm, which maps them to a (small) set of so-called master integrals, multiplied by rational functions in the dimension $d$. At this stage, we observe cancellation of the gauge-parameter dependence, serving as a check for the reduction. As an additional strong check, we use our implementation [19] of the strategy to solve a truncated set of IBP relations, based on lexicographic ordering of integrals [20].

The set of (massless, two-point) master integrals is known analytically in terms of Gamma functions, for generic dimension $d$, as given, e.g., in Ref. [8]. Expanding prefactors as well as master integrals about $d = 4 - 2\epsilon$ and renormalizing the gauge coupling, we again arrive at Eq. (6).

To conclude, we have evaluated the $\mathcal{O}(\alpha_s^2)$ correction to the colour-octet static potential using two independent techniques. Both evaluations are in agreement, giving us confidence in our main result, Eq. (6).

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